Riddler April 16, 2021:
The Case Of The Crescent Moon

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Question 1. *After a new moon, the crescent appears to grow slowly at first. At some point, the moon will be one-sixth full by area, then one-quarter full, and so on. Eventually, it becomes a half-moon, at which point its growth begins to slow down.*

How many times faster is the area of the illuminated moon growing when it is a half-moon versus a one-sixth moon?

(Some simplifying assumptions you might make for this problem are that the moon is a perfect sphere, that its orbit around Earth is a perfect circle, that the moon orbits the Earth much faster than the Earth orbits the sun and that the sun is very, very far away.)

For simplicity, suppose the Moon has unit radius and consider the spherical coordinate transformation for the paramterizing the surface of the moon:

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{aligned}$$

The *lunar terminator* is the division between the illuminated and dark hemispheres of the Moon. At a given Earth-Moon-Sun angle, the terminator will consist of the points on the Moon with constant azimuthal angle ϕ. When viewed from Earth, the terminator is effectively projected onto the xz-plane. So we first determine the equation of the terminator as projected onto this plane. For a fixed ϕ, we have $y^2 = \sin^2\theta\sin^2\phi = (1-\cos^2\theta)\sin^2\phi = (1-z^2)\sin^2\phi$ and thus

$$x^2 = 1 - z^2 - y^2 = (1-z^2)(1-\sin^2\phi) = (1-z^2)\cos^2\phi.$$

That is, the terminator projected onto the xz-plane is given by

$$x = \sqrt{1-z^2}\cos\phi.$$

It is now straightforward to compute the area of the illuminated portion of the disk for a given azimuthal angle of the terminator:

$$\begin{aligned} A &= \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}\cos\phi} \mathrm{d}x\,\mathrm{d}z \\ &= (1+\cos\phi)\pi. \end{aligned}$$

The rate of change of the illuminated area with respect to change in ϕ is

$$\frac{\mathrm{d}A}{\mathrm{d}\phi} = -\pi \sin\phi.$$

As the Moon orbits around the sun, we will assume that the angle of the terminator line changes at a constant rate (i.e., $\mathrm{d}\phi/\mathrm{d}t$ is constant). Now we want to know: for some fixed fraction $f \in [0,1]$ of area, what is the rate of change of the area (with respect to ϕ) when the area is equal to $A = 2\pi f$? We have $\cos\phi = 2f - 1$ and thus $\sin\phi = \sqrt{1-(2f-1)^2} = 2\sqrt{f(1-f)}$, hence

$$\left.\frac{\mathrm{d}A}{\mathrm{d}\phi}\right|_{A=2\pi f} = -2\pi\sqrt{f(1-f)}.$$

The answer to the question is therefore given by

$$\frac{\left.\frac{\mathrm{d}A}{\mathrm{d}\phi}\right|_{A=\frac{2\pi}{2}}}{\left.\frac{\mathrm{d}A}{\mathrm{d}\phi}\right|_{A=\frac{2\pi}{6}}} = \frac{\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)}}{\sqrt{\frac{1}{6}\left(1-\frac{1}{6}\right)}} = \frac{\frac{1}{2}}{\frac{1}{6}}\frac{1}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \approx 1.3416$$