Riddler December 10, 2021: Can You Win The Fencing Relay?

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Question 1. *You are the coach at Riddler Fencing Academy, where your three students are squaring off against a neighboring squad. Each of your students has a different probability of winning any given point in a match. The strongest fencer has a 75 percent chance of winning each point. The weakest has only a 25 percent chance of winning each point. The remaining fencer has a 50 percent probability of winning each point.*

The match will be a relay. First, one of your students will face off against an opponent. As soon as one of them reaches a score of 15, they are both swapped out. Then, a different student of yours faces a different opponent, continuing from wherever the score left off. When one team reaches 30 (not necessarily from the same team that first reached 15), both fencers are swapped out. The remaining two fencers continue the relay until one team reaches 45 points.

As the coach, you can choose the order in which your three students occupy the three positions in the relay: going first, second or third. How will you order them? And then what will be your team's chances of winning the relay?

Solution. Let p_1, p_2, and p_3 be the probabilities for the first, second, and third fencer on your team to win a given point. (That is, (p_1, p_2, p_3) is some permutation of $(3/4, 1/2, 1/4)$.) To start breaking down the problem, consider first only the first round of the relay. What is the probability our team (team A) wins the first round? The first fencer on Team A must win exactly 15 points and Team B wins at most 14 points. How many different ways can Team A win exactly 15 points and Team B win exactly b_1 points in the first round, where $b_1 \in \{0, 1, \ldots, 14\}$? Team B must win exactly b_1 of the first $14 + b_1$ points, then Team A must win the last point. Thus, the number of ways that this situation can occur is equal to

$$\binom{14 + b_1}{b_1}.$$

For each $b_1 \in \{0, 1, \ldots, 14\}$, the probability that Team A wins the first round while Team B wins exactly b_1 rounds is equal to

$$\Pr(A \text{ wins 15 points and } B \text{ wins } b_1 \text{ points in round 1}) = \binom{14 + b_1}{b_1} p_1^{15} (1 - p_1)^{b_1}.$$

Summing over all the different numbers of points that Team B can earn in the first round, we see that the probability that Team A wins the first round is equal to

$$\Pr(A \text{ wins first round}) = \sum_{b_1=0}^{14} \binom{14 + b_1}{b_1} p_1^{15} (1 - p_1)^{b_1}.$$

We'll need to build up a number of sums like this to compute the probability that Team A wins at the end of *three* rounds.

Consider now the different ways that Team A can win the first *two* rounds. If $b_1 \in \{0, 1, \ldots, 14\}$ is the number of points that Team B has after round 1, the possible number of points that Team B can have at the end of round 2 must satisfy $b_1 \leq b_2 \leq 29$. For a given $b_2 \in \{b_1, b_1 + 1, \ldots, 29\}$, the probability that Team A wins the first two rounds such that Team B has b_1 points at the end of the first round and b_2 points at the end of the second round is equal to

$$\binom{14 + b_1}{b_1}\binom{14 + b_2 - b_1}{b_2 - b_1} p_1^{15}(1 - p_1)^{b_1} p_2^{15}(1 - p_2)^{b_2 - b_1}.$$

Carrying on, suppose that Team A wins the first *three* rounds. If Team B has b_1 points after round 1, b_2 points after round 2, and b_3 points after round 3, then it must be the case that

$$0 \leq b_1 \leq 14, \quad b_1 \leq b_2 \leq 29, \quad \text{and} \quad b_2 \leq b_3 \leq 44,$$

and the probability that this occurs is equal to

$$\binom{14 + b_1}{b_1}\binom{14 + b_2 - b_1}{b_2 - b_1}\binom{14 + b_3 - b_2}{b_3 - b_2} p_1^{15} p_2^{15} p_3^{15}(1 - p_1)^{b_1}(1 - p_2)^{b_2 - b_1}(1 - p_3)^{b_3 - b_2}.$$

Summing over all of these possibilities gives us the probability that A wins the match by winning all three rounds of the relay:

$$\begin{aligned}
&\Pr(A \text{ wins all 3 rounds}) \\
&\quad = \sum_{b_1=0}^{14}\sum_{b_2=b_1}^{29}\sum_{b_3=b_2}^{44}\binom{14 + b_1}{b_1}\binom{14 + b_2 - b_1}{b_2 - b_1}\binom{14 + b_3 - b_2}{b_3 - b_2} \\
&\qquad\qquad \times p_1^{15} p_2^{15} p_3^{15}(1 - p_1)^{b_1}(1 - p_2)^{b_2 - b_1}(1 - p_3)^{b_3 - b_2}.
\end{aligned}$$

Similar analysis can be used to construct sums to compute the probabilities of other possible outcomes. For example, the probability that Team B wins the first round, but Team A wins the second and third rounds is equal to

$$\begin{aligned}
&\Pr(B \text{ wins first round and } A \text{ wins second and third round}) \\
&\quad = \sum_{a_1=0}^{14}\sum_{b_2=15}^{29}\sum_{b_3=b_2}^{44}\binom{a_1 + 14}{a_1}\binom{29 - a_1 + b_2 - b_1}{b_2 - b_1}\binom{14 + b_3 - b_2}{b_3 - b_2} \\
&\qquad\qquad \times p_1^{a_1} p_2^{30 - a_1} p_3^{15}(1 - p_1)^{15}(1 - p_2)^{b_2 - 15}(1 - p_3)^{b_3 - b_2}.
\end{aligned}$$

For Team A to win the relay, they *must* win the third round, but they could win or lose any combination of the first two rounds.

Writing program to compute all of these different sums and adding them together would be a bit tedious, but fortunately we can make use of recursion to simplfy things and sum up all of them at once. The following code snippet does exactly that!

Running the code yields that the optimal permutation of fencers is $(.25, .5, .75)$. That is, order your fencers in increasing ability—your worst fencer should go first, then your second best, and end with your best. The probability that your team wins the relay with this strategy is roughly 93.17%. Exactly, this probability is

$$\frac{21277034269564210496704281489241120051522347369}{22835963083295358096932575511191922182123945984} \approx 0.9317336077289661.$$

```
from scipy.special import comb
from fractions import Fraction
import itertools

N = 15 # incremental number of points to win a relay round
M = 3  # total number of increments of N that need to be won

p = [.25, .5, .75]

def df(A, B, p_cum, p_tot):
    g = max(A//N, B//N)
    if g == M:
        return (p_tot + p_cum) if A>B else p_tot

    # A wins next round.
    m = N*(g+1) - A # number of points won by A
    for n in range(N*(g+1) - B): # loop through all possible numbers of points that B can win
        pnew = comb(m+n-1,n,exact=True) * p[g]**m * (1-p[g])**n
        p_tot = df(A+m, B+n, p_cum*pnew, p_tot)

    # B wins next round.
    n = N*(g+1) - B # number of points won by B
    for m in range(N*(g+1) - A): # loop through all possible numbers of points that A can win
        pnew = comb(m+n-1,m,exact=True) * p[g]**m * (1-p[g])**n
        p_tot = df(A+m, B+n, p_cum*pnew, p_tot)

    return p_tot

best_strat, best_prob = [], 0
for strat in itertools.permutations(p):
    win_prob = df(0, 0, 1, 0)
    if win_prob > best_prob:
        best_prob = win_prob
        best_strat = strat

print(best_strat, best_prob)
```