Riddler January 7, 2022:
Can You Trek The Triangle?

Mark Girard

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Question 1. *Amare the ant is traveling within Triangle ABC, as shown below. Angle A measures 15 degrees, and sides AB and AC both have length 1. Angle A measures 15 degrees, and AB and AC both*

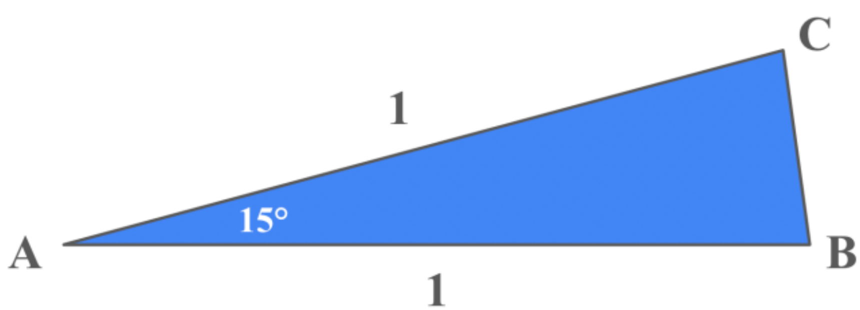


Figure 1: Isosceles triangle ABC with vertex A.

have length 1. Amare starts at point B and wants to ultimately arrive on side AC. However, the queen of his colony has asked him to make several stops along the way. Specifically, his path must:

- *Start at point B.*
- *Second, touch a point—any point—on side AC.*
- *Third, touch a point—any point—back on side AB.*
- *Finally, proceed to a point–any point—on side AC (not necessarily the same point he touched earlier). What is the shortest distance Amare can travel to complete the queen's desired path?*

What is the shortest distance Amare can travel to complete the queen's desired path?

Solution. Each valid path is uniquely determined by a triple (x, y, z) of numbers $x, y, z \in [0, 1]$, where x, y, and z indicate the distance from point A to the point on the corresponding line segment (AC or AB). See the figure below.

We can compute the lengths of each of these three new segments using hte Law of Cosines. With an angle of $\theta = 12^\circ = \frac{\pi}{12}$, applying the Law of Cosines we find that the lengths of each of

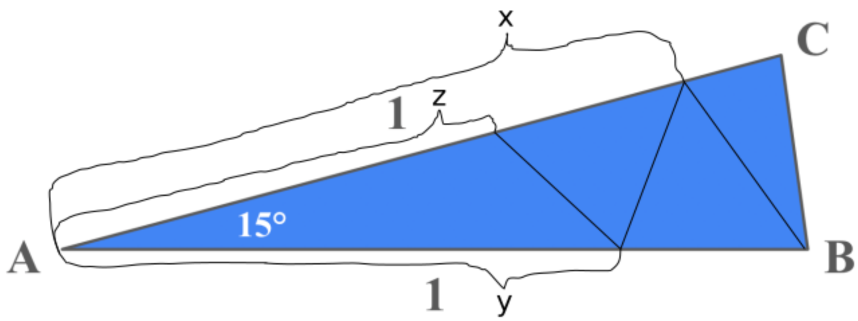


Figure 2: Isosceles triangle ABC with vertex A.

these segments are

$$\sqrt{1+x^2-2x\cos\theta}, \qquad \sqrt{x^2+y^2-2xy\cos\theta}, \qquad \text{and} \qquad \sqrt{y^2+z^2-2yz\cos\theta}.$$

Generalizing the problem For now, however, we can solve a more generalized version of this problem, in which the ant Amare needs to find the optimal path in a triangle whose interior angle is arbitrary (i.e., not necessarily $\theta = \frac{\pi}{12}$). We may allow $\theta \in (0\frac{\pi}{2})$ For simplicity, we define $\gamma = \cos\theta$, and we have $\gamma \in (0,1)$.

Trivial upper bound First, let's place an upper bound on the length of Amare's optimal path. Amare could simply take $x = y = z = 0$ and walk along tthe lower edge of the triangle from B to A. The point A is clearly on both line segments AC and AB, so this is a valid path and has a total length of 1.

Defining the objective Given a path defined by (x,y,z), the total length of the path traveled by Amare is equal to

$$f(x,y,z) = \sqrt{1+x^2-2x\gamma} + \sqrt{x^2+y^2-2xy\gamma} + \sqrt{y^2+z^2-2yz\gamma}. \tag{1}$$

The gradient of this function is given by

$$\nabla f(x,y,z) = \begin{pmatrix} \frac{x-\gamma}{\sqrt{1+x^2-2x\gamma}} + \frac{x-y\gamma}{\sqrt{x^2+y^2-2xy\gamma}} \\ \frac{y-x\gamma}{\sqrt{x^2+y^2-2xy\gamma}} + \frac{y-z\gamma}{\sqrt{y^2+z^2-2yz\gamma}} \\ \frac{z-y\gamma}{\sqrt{y^2+z^2-2yz\gamma}} \end{pmatrix}. \tag{2}$$

The optimal value must occur when the gradient is equal to zero. Setting the third component of the gradient in (2) to zero, we find that y and z must satisfy

$$z - y\gamma = 0. \tag{3}$$

Making the substitution $z = y\gamma$, the second component of the gradient in (2) reduces to

$$\frac{y - x\gamma}{\sqrt{x^2 + y^2 - 2xy\gamma}} + \sqrt{1 - \gamma^2}.$$

Equating this expression with zero, we find that

$$x\gamma - y = \sqrt{1 - \gamma^2}\sqrt{x^2 + y^2 - 2xy\gamma} \tag{4}$$

and squaring both sides yields

$$x^2\gamma^2 - 2xy\gamma + y^2 = (1 - \gamma^2)(x^2 + y^2 - 2xy\gamma)$$

or equivalently

$$(1 - 2\gamma^2)x^2 + 2\gamma^3 xy - \gamma^2 y^2 = 0.$$

It can be verified by expanding out the square that this is equivalent to

$$\gamma^2 \left(y - \frac{x}{\gamma} \right) \left(y - \frac{2\gamma^2 - 1}{\gamma} x \right) = 0.$$

It must therefore be the case that either

$$y = \frac{x}{\gamma} \qquad \text{or} \qquad y = \frac{2\gamma^2 - 1}{\gamma} x.$$

Note however from (4) that $x\gamma - y$ must be nonnegative, and thus $y \leq x\gamma$. Thus it cannot be the case that $y = x/\gamma$, as this would imply that $1/\gamma \leq \gamma$, which is a contradiction becauase $0 < \gamma < 1$. It follows that

$$y = \frac{2\gamma^2 - 1}{\gamma} x. \tag{5}$$

Making the substitution $y = \frac{2\gamma^2-1}{\gamma}x$, the first component of the gradient in (2) reduces to

$$\frac{x - \gamma}{\sqrt{1 + x^2 - 2x\gamma}} + 2\gamma\sqrt{1 - \gamma}$$

Equating this expression with zero, we find that

$$x - \gamma = 2\gamma\sqrt{1 - \gamma^2}\sqrt{1 + x^2 - 2x\gamma} \tag{6}$$

and squaring both sides yields

$$x^2 - 2x\gamma + \gamma^2 = 4\gamma^2(1 - \gamma^2)(1 + x^2 - 2x\gamma)$$

or equivalently

$$x^2 - 2\gamma x + \gamma^2 \frac{3 - 4\gamma^2}{(1 - 2\gamma^2)^2} = 0$$

(as long as $\gamma \neq 1/\sqrt{2}$). (If $\gamma \neq 1/\sqrt{2}$ then this has no solutions.) This is equivalent to

$$\left(x - \frac{\gamma}{2\gamma^2 - 1}\right)\left(x - \frac{\gamma\left(4\gamma^2 - 3\right)}{2\gamma^2 - 1}\right) = 0.$$

It must therefore be the case that either

$$x = \frac{\gamma}{2\gamma^2 - 1} \qquad \text{or} \qquad x = \frac{\gamma\left(4\gamma^2 - 3\right)}{2\gamma^2 - 1}.$$

However, note that $\frac{\gamma}{2\gamma^2-1} \notin [0,1]$ for all $\gamma \in (0,1) \setminus \{1/\sqrt{2}\}$. Because we must have $x \in [0,1]$, we must therefore have that

$$x = \frac{\gamma\left(4\gamma^2 - 3\right)}{2\gamma^2 - 1}.$$

Hence the only reasonable candidate for the optimal path is given by

$$(x, y, z) = \left(\frac{\gamma\left(4\gamma^2 - 3\right)}{2\gamma^2 - 1}, 4\gamma^2 - 3, \gamma\left(4\gamma^2 - 3\right)\right).$$

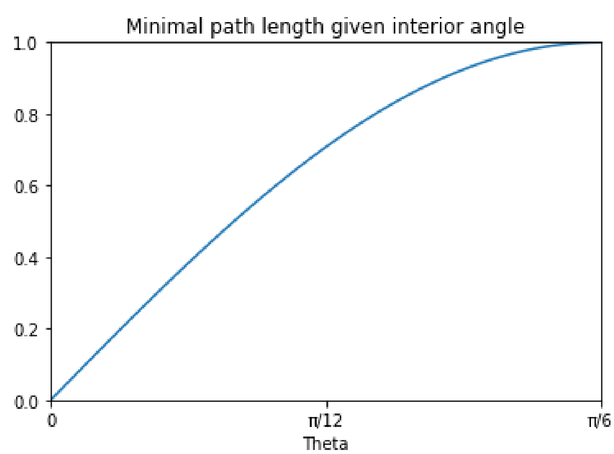
Because this path only makes sense if $x, y, z \in [0,1]$, we only have a non-trivial optimal minimal path in the case when $\gamma \in [\sqrt{3}/2, 1]$. Because $\gamma = \cos\theta$, this means that the angle BAC must satisfy $\theta \in [0, \frac{\pi}{6}]$. For γ in this range, plugging these values into the objective function in (1), we see that the length of this optimal path is equal to

$$f\left(\frac{\gamma\left(4\gamma^2 - 3\right)}{2\gamma^2 - 1}, 4\gamma^2 - 3, \gamma\left(4\gamma^2 - 3\right)\right) = \sqrt{1 - \gamma^2}\left(4\gamma^2 - 1\right).$$

(After a bit of algebra!) In terms of θ, this is

$$\left(4\cos^2\theta - 1\right)\sin\theta.$$

A plot of this optimal path length for a given interior angle is shown in the figure below.



Case when $\theta = \pi/12$ The specific case of the problem statement has $\theta = \theta = \pi/12$ and thus

$$\gamma = \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

The optimal path is described by

$$(x, y, z) = \left(\sqrt{\frac{2}{3}}, \sqrt{3}-1, \frac{1}{\sqrt{2}}\right) \approx (0.816, 0.732, 0.707)$$

and the length of this minimal path is equal to

$$(4\cos^2\tfrac{\pi}{12} - 1)\sin\tfrac{\pi}{12} = \frac{1}{\sqrt{2}} \approx 0.707.$$