Riddler July 1, 2022:
The Two (Astronomer's) Towers

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Riddler classic

Question 1. *As the Royal Astronomer of Planet Xiddler, you wake up from a dream in which you measured the planet's radius using a satellite. "How silly!" you think to yourself. "Satellites haven't even been invented yet!" And so you and another astronomer set out to investigate the curvature of the planet.*

The two of you climb two of the tallest towers on the planet, which happen to be in neighboring cities. You both travel 100 meters up each tower on a clear day. Due to the curvature of the planet, you can barely make each other out.

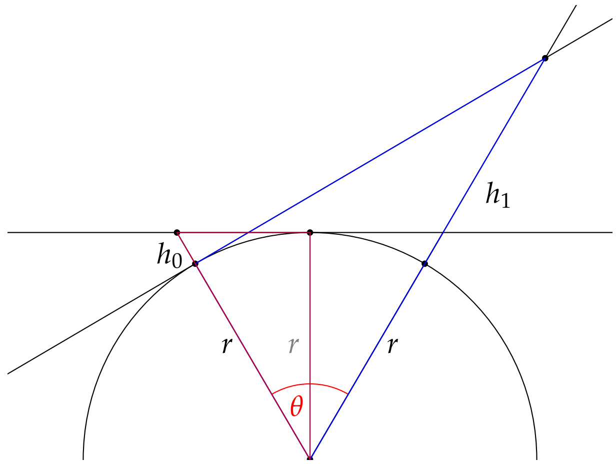
Next, your friend returns to the ground floor of their tower. How high up your tower must you be so that you can just barely make out your friend again?

Solution

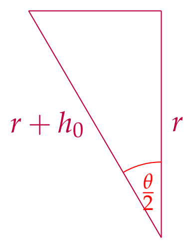
For this problem we will define the following variables:

- r is the radius of the planet,
- θ is the angle between the two towers,
- h_0 is the initial height to which the two astronomers climb in their towers (i.e., $h_0 = 100\,\mathrm{m}$), and
- h_1 is the taller height to which the second astronomer must climb to see the first astronomer at the base of the first tower.

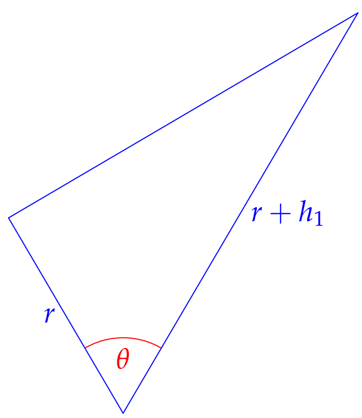
With these variables we have the following picture.



Initially, the two astronomers are each a height of h_0 above the ground at their towers. For the two astronomers to just barely see each other, the line segment connecting them must lie tangent to the planet at the midpoint between them. Taking the important bits of the geometry from this setup gives us the following right triangle.



After the first astronomer returns to the ground, the second astronomer climbs higher in the second tower to see his friend. Now, the line segment connecting the two astronomers must line tangent to the planet at the base of the first tower. This gives us the following right triangle.



Using basic trigonometry, we find that

$$\cos\frac{\theta}{2} = \frac{r}{r+h_0} \quad \text{and} \quad \cos\theta = \frac{r}{r+h_1}.$$

We can easily combine these two equations using the following trigonometric identity,

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

to find the relationship between h_0 and h_1:

$$\frac{r}{r+h_1} = 2\left(\frac{r}{r+h_0}\right)^2 - 1.$$

Finally, rearranging allows us to solve for h_1 in terms of r and h_0,

$$h_1 = r\left(\frac{1}{2\left(1+h_0/r\right)^{-2} - 1} - 1\right). \tag{1}$$

One question we might ask is, what is the relationship between h_0 and h_1 in the limit of a *really* large planet (i.e., as $r \to \infty$)? For this, we can take the limit of the expression for h_1 in (1) as $h_0/r \to 0$. Making use of the following derivative,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2\left(1+x\right)^{-2} - 1} - 1\right)\Bigg|_{x\to 0^+} &= \frac{-1}{\left(2\left(1+x\right)^{-2} - 1\right)^2}\left(-4(1-x)^{-3}\right)\Bigg|_{x\to 0^+} \\ &= 4\end{aligned}$$

we find that

$$\lim_{r\to\infty}\frac{h_1}{h_0} = 4.$$

Thus, if the radius of the home planet of our favourite astronomers is very large compared to 100 m, the second astronomer should expect to climb to a height of slightly above 400 m to see his friend.

On Earth

If our astronomers were on Earth, which has a radius of approximately 6.371 million meters (i.e., $r = 6.371 \times 10^6$ m), then the second astronomer must climb to a height of

$$h_1 \approx 400.016\,\mathrm{m}$$

(so, only about 1.6 cm above the 400 m mark). Moreover, the angle between the astromers is

$$\begin{aligned}\theta &= 2\arccos\left(\frac{r}{r+h_0}\right) \\ &\approx 0.0112\end{aligned}$$

and so the surface distance between the bases of the two towers is

$$r\cos\theta \approx 71.4\,\mathrm{km}.$$